Determination of the Separation Point in Laminar Boundary-Layer Flows

V. A. Wehrle*
Communications Research Center, Ottawa, Canada

An analytical scheme is presented for the determination of the separation point in laminar boundary-layer flows. Unlike conventional approaches the scheme does not require the full-field solution of the governing partial differential equation, but rather the solution of a first-order set of ordinary differential equations defined in the neighborhood of the leading edge. The development of the scheme is based upon the assumption, supported by examination of several numerical solutions from the published literature, that the value of a particular parameter of the boundary layer is invariant with respect to the nature of the external flow. This parameter is the slope, evaluated at the leading edge, of the curve for the normalized wall shear stress vs the normalized streamwise coordinate; its apparent value is $-\frac{2}{3}$. The scheme is demonstrated to compute accurately the separation points of several flows for which comparisons with previously published results are possible. Predictions are made for the separation points of some general classes of external flows for which complete solutions do not yet exist—including two that exhibit incipient separation. It is suggested that the determination of the separation point for incipient separation flows be adopted as a challenging test of accuracy in the evaluation of boundary-layer solution software packages.

Introduction

THE determination of the separation point in boundary-layer flows has been the subject of many investigations over the past few decades. The usual procedure is to apply numerical methods to the governing partial differential equation, compute the full-field solution, and thereby obtain the streamwise station at which the wall shear stress becomes zero. This solution procedure is not without its difficulties; it is well known that the wall shear stress approaches zero in a singular fashion at the separation point, a fact that invariably gives rise to problems of numerical convergence there. One of the earliest documentations of such numerical difficulties is by Hartree,[1] who sought to improve upon the accuracy of Howarth's[2] solution to the linearly retarded flow problem. Stimulated by Hartree's numerical evidence of the presence of a singularity at the separation point, Goldstein[3] undertook an analytical investigation of the nature of the singularity. Building on his earlier work[4] and making certain key (but as yet uncontested) assumptions, he developed a formal asymptotic solution for the flow in the immediate neighborhood of the separation point. He concluded that the wall shear stress tends to zero as the square root of the distance measured upstream from the separation point. While he acknowledged that "There is no mathematical proof that a solution exists with singularities of the type considered near separation...," he found it difficult to draw any other conclusions on the nature of the singularity given the assumptions he had made. A number of numerical solutions[5-11] to the separating laminar boundary-layer problem, performed since these key early contributions, confirm the difficulty of obtaining results in the immediate neighborhood of the separation point due to the singularity there. The subject of laminar separation and of singular flows and flows that are regular at separation has been reviewed by Brown and Stewartson.[12] Their assessment of the literature indicates "... that in general an incompressible boundary layer is singular at separation when the pressure gradient is prescribed." However, they point out that, from a theoretical point of view, it is not necessary that a singularity must occur. The specific question as to whether or not the singularity at separation is removable has been addressed by Stewartson,[13] who applied the method of matched asymptotic expansions to a triple-deck model. Within the limitations of the model, he concludes that a subsonic flow cannot remain analytic near separation and that, therefore, the singularity is not removable.

In the following sections, an analytical scheme, the basic features of which were previously introduced in Ref. 14, is developed for the determination of the separation point. The mathematical model is presented, a key assumption is corroborated, solutions are offered for several external flows (including two that exhibit incipient separation), and restrictions concerning the applicability of the scheme are discussed.

Mathematical Model

Consider the steady incompressible two-dimensional laminar boundary-layer flow past solid bodies for which the governing equation is

$$f''' + ff'' + \beta(1 - f'^2) = 2\xi(f'f'_\xi - f''f_\xi)$$
$$f(\xi,0) = f'(\xi,0) = 0 \qquad f'(\xi,\infty) = 1 \tag{1}$$

where

$$\xi = \frac{1}{L}\int_0^x \frac{u_e}{u_\infty}\left(\frac{r}{R}\right)^{2n} \mathrm{d}x \qquad \eta = \left(\frac{r}{R}\right)^n \frac{u_e y}{\sqrt{2\nu u_\infty L\xi}}$$

$$f = \int_0^\eta \frac{u}{u_e}\,\mathrm{d}\eta \qquad \beta = 2\xi(\ell n\, u_e)_\xi$$

Here, x and y are physical coordinates along the surface of the body and normal to it, respectively, with $x = 0$ at the leading edge and $y = 0$ at the body. The prime denotes the differentiation with respect to η and subscript ξ the differentiation with respect to ξ. L and R are reference lengths of the body, $n = 0$ for coplanar flows and $n = 1$ for axisymmetric

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*Staff Member, Space Mechanics Directorate.

flows. The subscripts e and ∞ refer to conditions at the local edge of the boundary layer and far upstream, respectively.

Consider now the streamwise coordinate transformation $\chi = (\beta_0 - \beta)/(\beta_0 - \beta_S)$, where the subscripts 0 and S refer to the streamwise stations at the leading edge and separation point, respectively. Equation (1) and the associated boundary conditions become

$$f''' + ff'' + [\beta_0 - (\beta_0 - \beta_S)\chi](1 - f'^2) = 2\lambda\chi(f'f'_\chi - f''f_\chi)$$
$$f(\chi, 0) = f'(\chi, 0) = 0, \qquad f'(\chi, \infty) = 1 \tag{2}$$

Here $\lambda = \xi[\ln(\beta_0 - \beta)]_\xi$ and is known from the external flowfield. Several things may be noted concerning the streamwise coordinate χ and the transformed Eq. (2). Regardless of the form of the external flow, χ always takes on the values 0 at the leading edge and 1 at the separation point. Also, the principal function β is always, by definition, a linear function of χ. Furthermore, the separation value of β or equivalently, through the known relationship $\beta(\xi)$, the physical location of the separation point appears explicitly in the governing equation. Finally, a new external flow parameter λ has been introduced and is a measure of the streamwise variation of β.

We perform a Maclaurin series expansion of the stream function f and the external flow parameter λ in terms of χ,

$$f(\chi, \eta) = \sum_{n=0}^{\infty} f_n(\eta)\chi^n \qquad \lambda(\chi) = \sum_{n=0}^{\infty} \lambda_n \chi^n$$

where

$$f_n(\eta) = \frac{1}{n!}\frac{\partial^n}{\partial\chi^n} f(\chi, \eta)\bigg|_{\chi=0} \qquad \lambda_n = \frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}\chi^n}\lambda(\chi)\bigg|_{\chi=0} \tag{3}$$

It follows that the mth derivative of $f(\chi, \eta)$ with respect to η is given by

$$f^{(m)}(\chi, \eta) = \sum_{n=0}^{\infty} f_n^{(m)}(\eta)\chi^n \tag{4}$$

and the derivative of $f^{(m)}(\chi, \eta)$ with respect to χ is given by

$$f_\chi^{(m)}(\chi, \eta) = \sum_{n=1}^{\infty} n f_n^{(m)}(\eta)\chi^{n-1} \tag{5}$$

Substituting Eqs. (3-5) into Eq. (2) and collecting terms of equal order in χ, we obtain a system of ordinary differential equations for the $f_n(\eta)$. The zeroth order equation is

$$f_0''' + f_0 f_0'' + \beta_0(1 - f_0'^2) = 0$$
$$f_0(0) = f_0'(0) = 0, \qquad f_0'(\infty) = 1 \tag{6}$$

which is of the Falkner-Skan type and yields the zeroth-order wall shear stress $f_0''(0)$ corresponding to the particular value of β_0. The first-order equation is

$$f_1''' + f_0 f_1'' + (1 + 2\lambda_0) f_1 f_0'' - 2(\beta_0 + \lambda_0) f_0' f_1'$$
$$-(\beta_0 - \beta_S)(1 - f_0'^2) = 0$$
$$f_1(0) = f_1'(0) = 0, \qquad f_1'(\infty) = 0 \tag{7}$$

Now the solution of this equation, which contains information on the location of the separation point explicitly through the parameter β_S, is indeterminate; we cannot solve for the first-order wall shear stress $f_1''(0)$ unless we know the value of β_S. Alternatively, if we can obtain information on the value of $f_1''(0)$ independently of the problem at hand, then we can solve Eq. (7) directly for β_S. In the following section, it is shown that this latter course of action is open to us. By examination of several numerical solutions from the published literature evidence is given that the value of the parameter

$$\frac{\mathrm{d}\bar{\tau}_w}{\mathrm{d}\chi}\bigg|_{\chi=0} \tag{8}$$

is invariant with respect to the nature of the external flow; the apparent value is $-\frac{2}{3}$. Here, $\bar{\tau}_w = f''(\chi, 0)/f''(0, 0)$ and is the wall shear stress normalized with respect to that at the leading edge. The significance of this observation is immediately clear; the boundary condition $f_1''(0)$ is precisely equal to this parameter multiplied by $f''(0, 0)$ so that, a priori, we are able to write $f_1''(0) = -\frac{2}{3}f_0''(0)$. Hence, the boundary conditions for Eq. (7) are overspecified and this enables the solution of the parameter β_S and, through the known external flow parameter $\beta(x)$, the physical location x_S of the separation point. As can be seen from the first-order system defined by Eqs. (6) and (7), the solution for β_S is dependent on the external flow only through the leading-edge parameter pair (β_0, λ_0).

The Parameter $\mathrm{d}\bar{\tau}_w/\mathrm{d}\chi|_{\chi=0}$

In Fig. 1 is shown a plot of $\bar{\tau}_w$ as a function of χ for the following eight external flows, for which solutions are available in the published literature: 1) potential flow past a circular cylinder,[8] 2) experimental flow past a circular cylinder,[9] 3) potential flow past a sphere,[9] 4) linearly retarded flow,[2] 5) quadratically retarded flow,[5] 6) quartically retarded flow,[5] 7) octally retarded flow,[5] and 8) experimental flow past an elliptic cylinder[15]. The close bandedness of the data points is intriguing, but of particular interest concerning the development of the present scheme is the slope of the data at $\chi = 0$ [i.e., the parameter defined by Eq. (8)]; it appears to have the value $-\frac{2}{3}$. The results for flow 8 plot somewhat erratically, especially near the leading edge. However, this is not surprising as Hartree made particular note of the difficulties he had encountered in representing Schubauer's observed pressure distribution there and in subsequently integrating the boundary-layer equations. It is to be noted also that since the coordinate χ is defined in terms of β_S, it follows that, to the

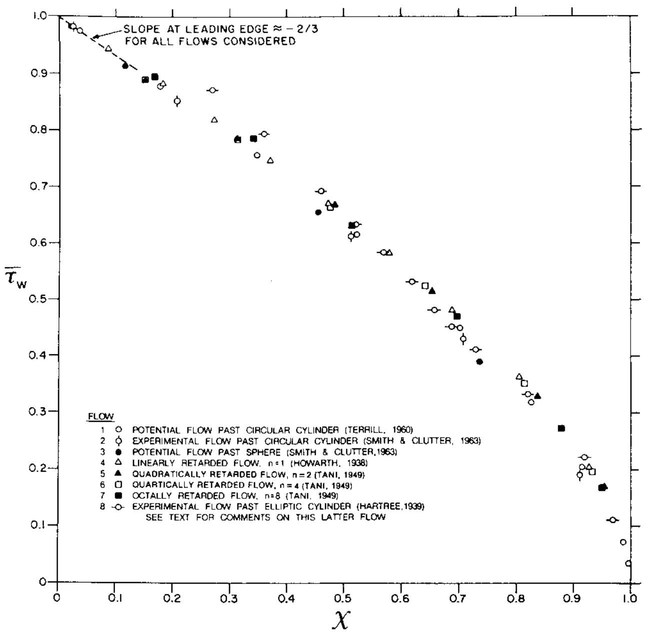

Fig. 1 Invariance of the parameter $\mathrm{d}\bar{\tau}_w/\mathrm{d}\chi|_{\chi=0}$.

extent each published solution for the location of the separation point in inexact, the track of the data points in Fig. 1 is also inexact.

In the following section, it is assumed that the parameter defined by Eq. (8) has the value $-\frac{2}{3}$ regardless of the nature of the external flow. The first-order system defined by Eqs (6) and (7), with overspecified boundary conditions, is employed to compute β_S for the flows shown in Fig. 1 and for some general classes of external flows.

Numerical Results

Table 1 gives a comparison of the presently predicted and previously published locations of the separation points for the eight flows considered in Fig. 1. The percentage difference, with respect to the previously published values, is remarkably small. Note that for flow 8 the present scheme predicts separation to occur at the physical location for which $\beta_S = -0.66395$; however, because Hartree did not define his modified pressure distribution for values of β less than -0.43, it is not possible to associate a physical location with the presently predicted value of β_S. We consider now the determination of the separation points for some general classes of external flows.

Symmetric Flow Past Elliptic Cylinders and Spheroids

Consider the ellipse defined in the xy plane with x axis intersects at -1 and $+1$ and y axis intersects at $-\epsilon$ and $+\epsilon$ and for which the equation is

$$x^2 + (y^2/\epsilon^2) = 1$$

The elliptic cylinder is created by an infinite extension of the ellipse in the $+z$ and $-z$ directions and the spheroid is created by a rotation of the ellipse about the x axis. If a uniform stream is approaching from the $-x$ direction the potential flow is given by

$$\frac{u_e}{u_\infty} = \frac{1+\epsilon P(\epsilon)}{\sqrt{1+\epsilon^2 x^2/(1-x^2)}} \qquad \beta = \frac{-2\epsilon^2 x Q(x)}{(1-x)\left[1-(1-\epsilon^2)x^2\right]}$$

where for cylinders

$$P(\epsilon) = 1 \qquad Q(x) = 1$$

and for spheroids

$$P(\epsilon) = \frac{\epsilon\left(\operatorname{sech}^{-1}\epsilon - \sqrt{1-\epsilon^2}\right)}{\sqrt{1-\epsilon^2} - \epsilon^2 \operatorname{sech}^{-1}\epsilon} \qquad Q(x) = \frac{\frac{2}{3} + x(1-x^2/3)}{(1+x)(1-x^2)}$$

It is easy to show that the parameter pair (β_0, λ_0) for cylinders is (1,1) and for spheroids is (0.5,0.5), regardless of the value assigned to ϵ. The present scheme yields the solutions $\beta_S = -0.66395$ for cylinders and $\beta_S = -0.43901$ for spheroids. Through the known external flow parameter $\beta(x,\epsilon)$, we can compute for any value of ϵ the separation coordinate x_S or, from the geometric relationship,

$$\phi = \pi - \cos^{-1}\frac{x}{\sqrt{\epsilon^2 + (1-\epsilon^2)x^2}}$$

the separation angle ϕ_S. The results are shown in Fig. 2 together with the solution by Terrill[8] for the circular cylinder and by Smith and Clutter[9] for the sphere.

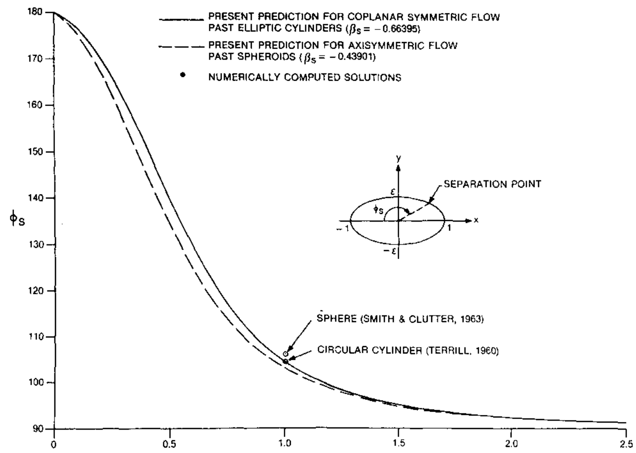

Fig. 2 Separation: elliptic cylinders and spheroids.

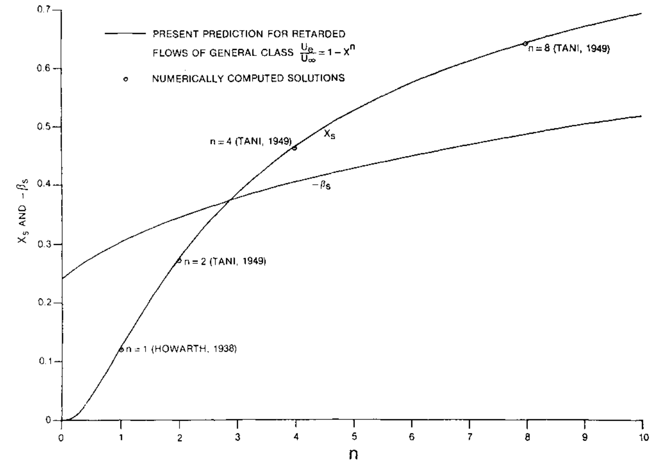

Fig. 3 Separation: Howarth-Tani flows.

Table 1 Comparison of presently predicted and previously published separation points

Flow no.	Flow description	Given (β_0, λ_0)	Computed β_S	Physical separation point: Presently predicted	Previously published	Ref.	Diff. from prev. pub., %
1	Potential flow past circular cylinder	(1,1)	−0.66395	104.432 deg	104.45 deg	8	0.017
2	Experimental flow past circular cylinder	(1,1)	−0.66395	78.847 deg	≈ 80 deg	9	1.4
3	Potential flow past sphere	(0.5,0.5)	−0.43901	102.915 deg	105.9 deg	9	2.8
4	Linearly retarded flow, $n = 1$	(0,1)	−0.30325	0.12404	0.120	2	3.4
5	Quadratically retarded flow, $n = 2$	(0,2)	−0.34474	0.27488	0.271	5	1.4
6	Quartically retarded flow, $n = 4$	(0,4)	−0.40437	0.46411	0.462	5	0.46
7	Octally retarded flow, $n = 8$	(0,8)	−0.48586	0.64166	0.640	5	0.26
8	Experimental flow past elliptic cylinder	(1,1)	−0.66395	—	1.98	15	—

Retarded Flows of the Howarth-Tani Type

For this class of retarded flows the external parameters are given by

$$\frac{u_e}{u_\infty} = 1 - x^n \qquad \beta = \frac{-2nx^n[1 - x^n/(n+1)]}{(1-x^n)^2}$$

where we shall consider values of $n > 0$. It is easy to show that $(\beta_0, \lambda_0) = (0, n)$ and the present scheme yields a different value of β_S for each value of n. Through the known external flow parameter $\beta(x, n)$ the location of the physical separation point can be determined. The results are shown in Fig. 3, together with the solutions of Howarth[2] for $n = 1$ and Tani[5] for $n = 2, 4$, and 8.

Retarded Flows of the Görtler Type

For this class of retarded flows, the external parameters are given by

$$\frac{u_e}{u_\infty} = [1 - S(n)x]^n$$

$$\beta = \frac{2n}{n+1}\left[1 - \frac{1}{[1 - S(n)x]^{n+1}}\right], \qquad n \neq -1$$

$$\beta = -2\ell n(1 + x), \qquad n = -1$$

where $S(n)$ stands for sign of n (this formulation ensures retarded flow whatever the sign of the exponent n). Regardless of the value of n, $(\beta_0, \lambda_0) = (0, 1)$ so that the solution for

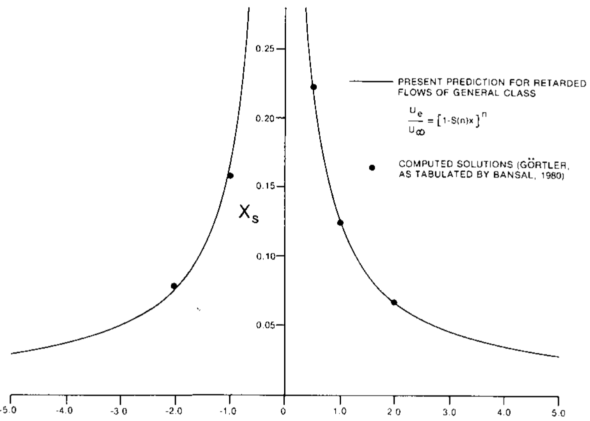

Fig. 4 Separation: Görtler flows.

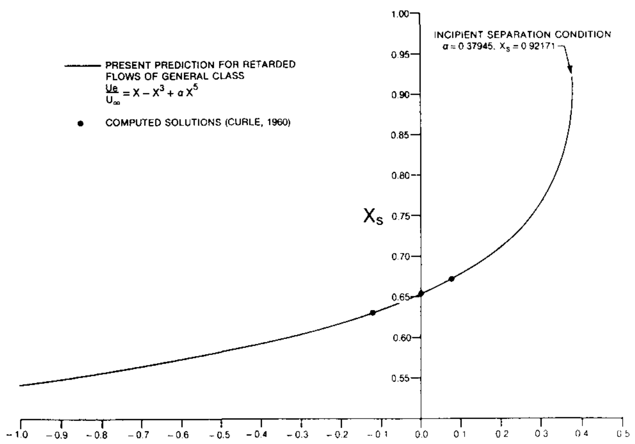

Fig. 5 Separation: Curle flows.

β_S by the present scheme is the same as that for Howarth's linearly retarded flow ($n = 1$), i.e. $\beta_S = -0.30325$. Through the known external flow parameter $\beta(x, n)$, the location of the physical separation point can be determined for any value of n. The results are shown in Fig. 4 together with the solutions of Görtler for $n = \frac{1}{2}$, 1, 2, -1, and -2 as tabulated by Bansal.[16] Not shown in Fig. 4 because of the scale of the plot, the present scheme yields $x_S \to 1$ as $n \to 0$ from above and $x_S \to \infty$ as $n \to \beta_S/(2 - \beta_S)$ (≈ -0.13166) from below. In the range $\beta_S/(2 - \beta_S) < n < 0$, the flow remains attached for all finite positive x.

Retarded Flows of the Curle Type: Incipient Separation

For this class of flows, the external parameters are given by

$$u_e/u_\infty = x - x^3 + \alpha x^5$$

$$\beta = \frac{(1 - x^2/2 + \alpha x^4/3)(1 - 3x^2 + 5\alpha x^4)}{(1 - x^2 + \alpha x^4)^2}$$

Regardless of the value of the parameter α, $(\beta_0, \lambda_0) = (1, 1)$ and the present scheme yields the solution $\beta_S = -0.66395$. Through the known external flow parameter $\beta(x, \alpha)$, the location of the physical separation point can be determined for any value of α. The results are shown in Fig. 5 together with the solutions of Curle[17] for $\alpha = -0.12156$, 0.0, and 0.07885. The results from the present scheme reveal that Curle's family of retarded flows contains a member that exhibits incipient separation, for which $\beta_{\min} = \beta_S$; this occurs for the particular value $\alpha_i = 0.37945$ at the streamwise station $x_{S,i} = 0.92171$. For values $\alpha > \alpha_i$ separation does not occur at all, whereas for values $\alpha < \alpha_i$ separation occurs with decreasing sensitivity of the solution for x_S to the value of α.

Because Curle's family of external flows includes a member exhibiting incipient separation, it is very much of theoretical interest. However, the family does not represent external flows past physical bodies of practical interest. In the following section, we consider the flow past a new family of symmetric airfoils that not only includes a member exhibiting incipient separation, but, in addition, represents a family of streamlined physical bodies which may be of practical interest in some applications.

Flow Past a New Family of Symmetric Airfoils: Incipient Separation

Consider the interaction of a uniform stream of strength u_∞ and a source of strength Q, the uniform stream approaching from the $+p$ direction and the source located at the origin of the $\boldsymbol{r}$ plane where $\boldsymbol{r} = p + iq$ (see sketch in Fig. 6). There is algebraic simplification, but no loss in generality of the problem if we specify that $Q/(2\pi u_\infty) = 1$. Then, the zero streamline, which may be imagined as the outer surface of a solid body deflecting the uniform stream in the same manner as does the source at the origin, is given by $\rho = \theta/\sin\theta$, where the polar coordinates ρ and θ are defined by $\boldsymbol{r} = \rho\exp(i\theta)$. The leading edge of the supposed body is located on the p axis at a distance of $+1$ from the origin and the half-thickness of it asymptotes to the line $q = \pi$. In the $\boldsymbol{r}$ plane the external velocity ratio on the surface of the body is given by

$$\hat{u}]_r = (1/\rho)\sqrt{1 - 2\rho\cos\theta + \rho^2}$$

where $\hat{u} = u_e/u_\infty$. Consider now the analytic transformation

$$z = \boldsymbol{r} - \ell n\frac{\boldsymbol{r} + k}{1 + k}$$

where k is a constant. If k is real positive† the exterior

†If k is real negative, the exterior surface in the $\boldsymbol{r}$ plane transforms to an interior surface in the z plane; if k is complex, airfoils with camber as well as thickness result.

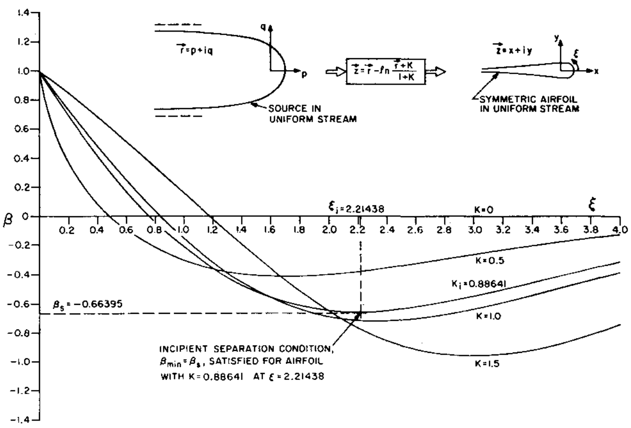

Fig. 6 Incipience: Wehrle's symmetric airfoil.

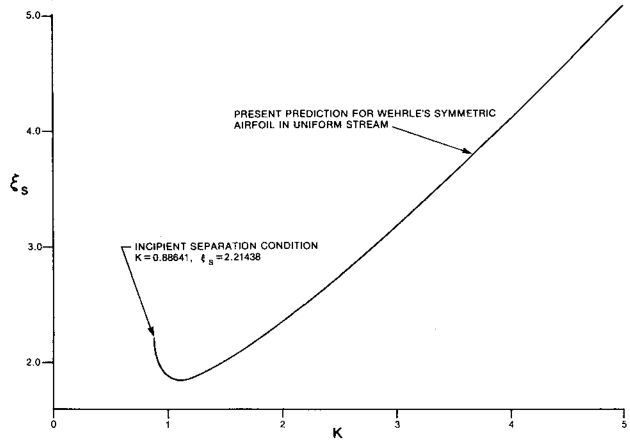

Fig. 7 Separation: Wehrle's symmetric airfoil.

surface of the supposed body in the r plane transforms to the exterior surface of a symmetric airfoil-shaped body of infinite chord in the z plane. The coordinates of the airfoil, defined by $z = x + iy$, are given by

$$x = \rho\cos\theta - \ln\left[\frac{1}{1+k}\sqrt{(\rho\cos\theta + k)^2 + \theta^2}\right]$$

$$y = \theta - \tan^{-1}\frac{\theta}{\rho\cos\theta + k}$$

and the particular coordinates locating the maximum thickness of the airfoil can be determined from the condition $\partial y/\partial\theta = 0$, which yields the maximum thickness relation $\rho\cos\theta = (1-k)/2$. After some algebra, the following parameters are defined in the z plane:

$$\hat{u}\rfloor_z = \frac{\hat{u}\rfloor_r}{|dz/dr|} = \sqrt{1 + \frac{k}{\rho^2}\frac{A}{B}}$$

$$\xi = \frac{1}{L}\int_0^{\ell}\hat{u}\rfloor_z d\ell = 1 - \rho\cos\theta + \ln\rho$$

$$\beta = \frac{2\xi}{\hat{u}\rfloor_z}\frac{d\hat{u}\rfloor_z}{d\xi} = \frac{2\xi}{\rho^2}\frac{\hat{u}\rfloor_z^2 - 1}{\hat{u}\rfloor_z^2\hat{u}_r\rfloor^2}\left(\rho\cos\theta - \frac{C}{A} - \frac{D}{B}\right)$$

In the evaluation of ξ, L has been taken as unity and an element of length on the airfoil is given by $d\ell^2 = dx^2 + dy^2$. Note that although the terms $\hat{u}\rfloor_z$ and $d\ell$ appearing in the definition of ξ are functions of the transformation parameter k, ξ itself turns out to be invariant with respect to k (it is shown in the following, however, that the particular value ξ_S is very much a function of k). The values for A, B, C, and D are given by

$$A = k + 2(1-k)\rho\cos\theta - 2\rho^2\cos 2\theta$$

$$B = (1-k)^2 - 2(1-k)\rho\cos\theta + \rho^2$$

$$C = k + (1-k)\rho^2 + (1 - k - 2\rho^2)\rho\cos\theta$$

$$D = (2-k)\rho^2 - (1 - k + \rho^2)\rho\cos\theta$$

Regardless of the value of the transformation parameter k the parameter pair (β_0, λ_0) takes on the value (1,1) and the present scheme yields $\beta_S = -0.66395$. Figure 6 shows a plot of the known external flow parameter $\beta(\xi, k)$ and it is seen that the incipient separation condition, for which $\beta_{\min} = \beta_S$, occurs for the airfoil with the particular value $k_i = 0.88641$ at the streamwise station $\xi_{S,i} = 2.21438$. For airfoils with $k < k_i$, separation does not occur at all (this suggests the use of such foils, made finite by truncation where the trailing edge is appropriately thin, in symmetric flow applications for low form drag), whereas for airfoils with $k > k_i$ separation occurs at the most upstream station for which $\beta = \beta_S$. Figure 7 shows a plot of ξ_S vs k; it is interesting to note that there occurs a minimum in ξ_S (equal to 1.85557 at $k = 1.0969$) and that, for large k, ξ_S appears to take on the asymptotic behavior $\xi_S = k$.

Discussion

Table 1 and Figures 2-5 give strong evidence, by way of comparison with previously published results, of the validity of the present scheme. Note that the solution for the separation point is entirely enabled by the first-order system defined by Eqs. (6) and (7); i.e., higher-order systems, obtained by substitution of Eqs. 3-5 with $n > 1$, contribute no additional information to the computation of β_S (and therefore of x_S). Higher-order systems of equations do, of course, contribute to the refinement of the solution for various field variables of interest, such as wall shear stress distribution and separation velocity profile. Indeed, it is intended to show in a future publication that one can pursue an extension of the current analytical treatment to obtain series solutions for the field variables which converge rapidly to the correct solutions, the convergence remaining rapid even at the separation station.

The application of the scheme is restricted to those external flows for which λ is an analytic function of χ in the domain of the attached flow, a restriction that follows from the requirements for analyticity in the Maclaurin series expansions of Eq. (3). In this connection, we observe from the definition of χ that for those flows in which the principal function β first *increases* in value with distance from the leading edge before subsiding to the separation value, the coordinate χ first *goes negative* from zero at the leading edge and then passes through zero again before reaching unity at the separation point. An examination of the definition of the parameter λ shows that it becomes singular at the second occurrence of $\chi = 0$, thereby rendering invalid the application of the present scheme to such flows. Of the flows considered in this paper, λ becomes singular in this manner only for symmetric flow past elliptic cylinders with $\epsilon > 2/\sqrt{3}$ and past spheroids with $\epsilon > \sqrt{6/5}$. Hence the accuracy of the predictions shown in Fig. 2 is questionable for values of ϵ greater than these respective limits, although the asymptotic trend $\phi_S \to 90$ deg. as $\epsilon \to \infty$ is certainly to be expected. Another flow for which the requirement for the analyticity of λ in the domain of the attached flow is not fulfilled is the incompressible flow past the semi-infinite parabola at angle of attack studied by Werle and Davis.[11] For this flow, λ becomes singular in the manner herein described for *all* nonzero angles of attack and therefore the present scheme cannot validly be applied to this problem. An interesting twist concerning analyticity occurs for the Curle family of flows; it can be shown that in the subfamily of separating flows (i.e., for $\alpha < \alpha_i$) a singularity in λ *always* arises regardless of the value of α, but that in every case it arises *beyond* the streamwise location of the separation point. Hence, for this family of flows, the application of the present scheme remains valid.

Riley and Stewartson[18] have examined the case $n \ll 1$ of the Görtler family of flows, which they argued represents reasonably well the external flow in the neighborhood of the trailing edge of slender aerodynamic shapes terminating in a wedge with included angle $2\pi n$. Applying the method of matched asymptotic expansions to a triple-deck model they found that $1 - x_S$ is of order $n^{3/2}$. The present scheme yields the global result

$$1 - S(n)\,x_S = \left[\frac{2n}{2n - (n+1)\beta_S}\right]^{1/(n+1)}$$

where β_S is a constant (≈ -0.30325) and, for $n \to 0$ from above, gives the limiting behavior $1 - x_S = -2n/\beta_S$. The failure of the present scheme to yield the $\mathcal{O}(n^{3/2})$ limiting behavior predicted by Riley and Stewartson[18] is attributed to the fact that the present scheme takes no account of the transition of the flow structure, in the immediate vicinity of the separation point, from that of the classical boundary-layer double deck to that of a triple deck. For this reason also, the uncertainty in the value of x_S as predicted by the present scheme is expected to be of the order of the streamwise extent of the triple-deck flow structure. For the case under discussion, this extent is $\mathcal{O}(n^{3/2})$ where $n \ll 1$ and compared to $x_S = \mathcal{O}(1)$, represents a very small uncertainty indeed.

For incipient separation flows, the adversity of the pressure gradient is *just sufficient* to cause separation to occur at all. The location of the separation point is extremely (in the theoretical limit, infinitely) sensitive to small variations in the governing free parameter of the external flow. Thus, it is anticipated that the determination of the separation point for incipient separation flows will present a challenging test of accuracy for boundary-layer solution software packages. Two such problems have been solved using the analytical scheme described herein and the solutions for the separation point so obtained are believed to be accurate and therefore valuable as reference solutions.

In the introductory section to their review paper on laminar separation Brown and Stewartson[12] suggest that the phenomenon of separation "...is closely related to the boundary layer upstream of S. Such an intimate relation has not of course been established beyond a shadow of doubt, either experimentally or theoretically. However the contrary view, that separation and boundary layer are unrelated, leaves the breakaway problem as a complete mystery, with no clue to its resolution given by experiment or theory. The view we adopt here is that the two are closely linked...." The present scheme demonstrates not only that the two are closely linked, but that the separation point is in fact entirely determinable by an analytical model defined in the neighborhood of the leading edge alone.

Conclusions

The main conclusions to be drawn from this paper are:

1) The value of a particular parameter of the boundary-layer flow is invariant with respect to the nature of the external flow. It is defined by Eq. (8), and has the value $-\frac{2}{3}$.

2) An analytical scheme, involving the particular parameter, enables the accurate determination of the separation point from considerations of the flow in the neighborhood of the leading edge alone.

3) Boundary-layer solution software packages can be subjected to a challenging test of accuracy by solving incipient separation flow problems. Two such problems have been solved using the analytical scheme described herein and the solutions for the separation point so obtained are believed to be accurate and therefore valuable as reference solutions.

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